International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Quantum Logics and Instruments

Sylvia Pulmannová[1]

Received July 4, 1997

In a combined quantum logic and convexity approach, an abstract notion of an instrument (state transformer) is introduced to describe quantum measurements. Some important classes of instruments (first kind, repeatable, ideal, Lüders) and relations among them are investigated.

INTRODUCTION

In the quantum logic approach, a generalization of classical probability theory is considered where the set of all random events of a quantum experiment is modeled by a quantum logic, i.e., an orthomodular σ-lattice, replacing the σ-field of subsets in the classical Kolmogorovian approach. In Pulmannová (1993, 1994, 1995), a theory of quantum measurements is formulated in analogy with the traditional Hilbert space approach (Busch *et al.*, 1991, 1996). The coupled physical system consisting of a measured object and a measuring apparatus is described in terms of a Boolean power of a quantum logic and a Boolean algebra. In this frame, the notion of a measuring instrument (see, e.g., Davies, 1976) can be formulated in which all information about the measurement is contained. It turns out that for a more detailed development of the measurement theory, quantum logics with some special properties are needed. For example, in Pulmannová (1994, 1995) so-called u-spectral logics are considered. Some authors (e.g., Abbati and Manià, 1984; Rüttimann, 1985) use a combination of quantum logic and convexity approaches. Following them, in this paper we consider quantum logics (orthomodular σ-lattices) which appear in the noncommutative spectral theory investigated by Alfsen and Shultz (1976). In this frame, we introduce an abstract definition of a measuring instrument and study instruments with some

[1] Mathematical Institute, Slovak Academy of Sciences, SK-814 73 Bratislava, Slovakia; e-mail: pulmann@mau.savba.sk.

0020-7748/98/0100-0163$15.00/0 © 1998 Plenum Publishing Corporation

special properties important from the point of view of physical experiments. Relations between u-spectral logics and the Alfsen and Shultz approach are studied in Pulmannová (n.d.).

1. BASIC DEFINITIONS AND RESULTS

Assume that X, Y are two positively generated ordered vector spaces. We say that X, Y are in *separating order duality* with respect to a bilinear form (1) if for $x \in X$, $y \in Y$,

$$x \geq 0 \Leftrightarrow \langle x, y\rangle \geq 0 \qquad \text{for all} \quad y \geq 0$$

$$y \geq 0 \Leftrightarrow \langle x, y\rangle \geq 0 \qquad \text{for all} \quad x \geq 0 \tag{1.1}$$

In what follows, we shall consider an order-unit space (A, e) and a base-norm space (V, K) (for definitions see, e.g., Alfsen, 1971, Ch. II, §1), and we assume that they are in separating order and norm duality, i.e., we assume together with (1.1) the following requirement, in which $a \in A$, $\rho \in V$:

$$\|a\| \leq 1 \Leftrightarrow |\langle a, \rho\rangle| \leq 1 \qquad \text{whenever} \quad \|\rho\| \leq 1$$

$$\|\rho\| \leq 1 \Leftrightarrow |\langle a, \rho\rangle| \leq 1 \qquad \text{whenever} \quad \|a\| \leq 1 \tag{1.2}$$

According to Alfsen and Shultz (1976, §7.1), an order-unit space (A, e) and a base-norm space (V, K) are in *spectral duality* if (1.1) and (1.2) are satisfied, A is pointwise monotone σ-complete, and if for every $a \in A$ and $\lambda \in \mathbb{R}$ there exists a projective face[2] F which is bicompatible with a,[3] and satisfies $a \leq \lambda$ on F, $a > \lambda$ on $F^{\#}$.

Let (A, e) be an order-unit space and (V, K) a base-norm space in spectral duality. Recall that an element $a \in A$ is a *projective unit* if it is an extreme point of the interval $[0, e]$. Under the above suppositions, the set $\mathcal{U}$ of all projective units forms an orthomodular σ-complete lattice.

A family $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ of projective units is said to be a *spectral family* if for $\lambda, \mu \in \mathbb{R}$

(i) $e_\lambda \leq e_\mu$ when $\lambda \leq \mu$,
(ii) $e_\lambda = \wedge_{\mu>\lambda} e_\mu$,
(iii) $\wedge_{\lambda\in\mathbb{R}} e_\lambda = 0$, $\vee_{\lambda\in\mathbb{R}} e_\lambda = e$.

A spectral family $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ has a *compact support* if there exist $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, such that $e_\lambda = 0$ for all $\lambda \leq \alpha$ and $e_\lambda = e$ for all $\lambda \geq \beta$.

A spectral family $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ is said to be a *spectral resolution* for a given element $a \in A$ if for every $\lambda \in \mathbb{R}$,

[2] A face F of K is projective if $F = (imP_F^*) \cap K$ for some P-projection P_F on A.
[3] i.e., compatible will all $b \in A$ which are compatible with a, where F and b are compatible if $P_F\, b \leq b$.

$$\rho \in K, \qquad \langle e_\lambda, \rho \rangle = 1 \Rightarrow \langle a, \rho \rangle \leq \lambda$$

$$\rho \in K, \qquad \langle e_\lambda, \rho \rangle = 0 \Rightarrow \langle a, \rho \rangle > \lambda$$

If (A, e) and (V, K) are in spectral duality, there is a 1–1 correspondence between elements $a \in A$ and spectral families $\{e_\lambda\}$ of compact supports, given by

$$a = \int \lambda \, de_\lambda$$

where the right side is a norm-convergent Riemann–Stiltjes integral. Moreover, a functional calculus can be introduced: If a is an element of A with spectral resolution $\{e_\lambda\}$ and if ϕ is a bounded Borel function of real variable, then there exists a unique element b of A such that for all $\rho \in K$,

$$\langle b, \rho \rangle = \int \phi(\lambda) \, d\langle e_\lambda, \rho \rangle$$

In particular, for every Borel set E the element p_E of A defined by

$$\langle p_E, \rho \rangle = \int_E d\langle e_\lambda, \rho \rangle \qquad \text{for} \quad \rho \in K$$

is a projective unit which belongs to the bicommutant of a. Moreover, the mapping $E \mapsto p_e$ from the Borel sets into $\mathcal{U}$ satisfies

(i) $P_{\mathbb{R}} = e,$
(ii) $p_E = \Sigma_n p_{En}$ for any disjoint sequence $\{E_n\}$ with $\cup_n E_n = E.$

In other words, the set $\mathcal{U}$ of projective units can be interpreted as a quantum logic, the set K can be identified with a rich[4] family of states on $\mathcal{U}$, and the set A with the set of bounded observables on $\mathcal{U}$. The expectation of an observable $a \in A$ in a state $\rho \in K$ is then $\rho(a) = \langle a, \rho \rangle$.

2. INSTRUMENTS

From the separating order and norm duality of V and A it follows that if $T\colon V \to V$ is a weakly continuous mapping, then for any fixed $a \in A$, $\rho \mapsto \langle a, T(\rho) \rangle$ is a weakly continuous linear functional on V, hence there is a unique element $T^*(a)$ of A such that $\langle a, T(\rho) \rangle = \langle T^*(a), \rho \rangle$, $\forall \rho \in V$, and $T^*\colon A \to A$ is a weakly continuous linear transformation of A. If, moreover, T is positive, i.e., $T(V^+) \subset V^+$, then T^* is positive as well. In what follows,

[4] A family M of states on a quantum logic L is rich if $a \not< b$ implies that $\exists m \in M$, $m(a) = 1$, $m(b) < 1$. In our setting, this condition holds due to Lemma 2.16(v) in Alfsen and Shultz (1976).

$L^+(V, V)$ and $L^+(A, A)$ denote the spaces of all weakly continuous positive linear transformations of V and A, respectively.

Definition 2.1. Let (V, K) and (A, e) be in spectral duality and $(\Omega, \mathcal{S})$ be a measurable space. An *instrument* (or *state transformer*)[5] is a mapping $I: \mathcal{S} \to L^+(V, V)$ such that:

(Ii) $\langle e, I(\Omega)(\rho)\rangle = \langle e, \rho\rangle \ \forall \rho \in V$,

(Iii) $I(\cup_{n=1}^{\infty} E_n)(\rho) = \Sigma_{n=1}^{\infty} I(E_n)(\rho) \ \forall \rho \in V$, where the sum converges in the weak sense for any disjoint sequence $(E_n) < \mathcal{S}$.

Definition 2.2. Let (V, K) and (A, e) be in spectral duality and $(\Omega, \mathcal{S})$ be a measurable space. A *dual instrument* (*dual state transformer*) is a mapping $I^*: \mathcal{S} \to L^+(A, A)$ such that:

(DIi) $I^*(\Omega)(e) = e$,

(DIii) $I^*(\cup_n{}^{n}{}_{=1} E_n)(a) = \Sigma_{n=1}^{\infty} I^*(E_n)(a) \ \forall a \in A$, where the sum converges in the weak sense for any disjoint sequence $(E_n) < \mathcal{S}$.

Theorem 2.3. To every instrument there corresponds a unique dual instrument, and conversely, to every dual instrument there corresponds a unique instrument.

Proof. (1) Let I be an instrument. For any fixed $a \in A$ and $E \in \mathcal{S}$, $\rho \mapsto \langle a, I(E)(\rho)\rangle$ is a weakly continuous positive functional on V, therefore there is a (unique) element $I^*(E)(a)$ in A such that

$$\langle a, I(E)(\rho)\rangle = \langle I(E)^*(a), \rho\rangle$$

and $I(E)^*: A \to A$ is a weakly continuous linear transformation. By (Ii), $\langle e, \rho\rangle = \langle e, I(\Omega)(\rho)\rangle = \langle I(\Omega)^*(e), \rho\rangle \ \forall \rho \in V$, hence $I(\Omega)^*(e) = e$. If $\{E_n\}$ is a disjoint sequence of elements of $\mathcal{S}$, by (Iii), for all $\rho \in V$,

$$\langle a, I\left(\bigcup_{n=1}^{\infty} E_n\right)(\rho)\rangle = \sum_{n=1}^{\infty} \langle a, I(E_n)(\rho)\rangle \qquad \forall a \in A$$

hence

$$\langle I\left(\bigcup_{n=1}^{\infty} E_n\right)^*(a), \rho\rangle = \sum_{n=1}^{\infty} \langle I(E_n)^*(a), \rho\rangle \qquad \forall \rho \in V$$

and hence $I(\cup_{n=1}^{\infty} E_n)^*(a) = \Sigma_{n=1}^{\infty} I(E_n)^*(a)$ for every $a \in A$. Putting $I^*(E) = I(E)^*$ for $E \in \mathcal{S}$, we obtain the desired dual instrument.

[5] In Busch *et al.* (1996) the notion *state transformer* replaces the commonly used notion *instrument* for the reason that it is a mathematical notion rather than a real instrument.

(2) Using the above arguments the other way round, we obtain that to every dual instrument there is a unique instrument. ■

Semiobservables on quantum logics have been investigated in Pulmannová (1980) as analogues of POV-measures (i.e., positive-operator-valued measures) in the Hilbert space approach (see, e.g., Berberian, 1966). They generalize the notion of an observable (a PV-measure, i.e., projection-valued measure in the Hilbert space approach).

Definition 2.4. Let (V, K) and (A, e) be in spectral duality and $(\Omega, \mathcal{S})$ a measurable space. A *semiobservable* is a mapping $X: \mathcal{S} \to \mathcal{E}$, where we put $\mathcal{E} = [0, e] \subset A$, such that:

(i) $X(\Omega) = e$,
(ii) $X(\cup_{n=1}^{\infty} E_n) = \Sigma_{n=1}^{\infty} X(E_n)$ for any sequence $\{E_n\}$ of disjoint sets from $\mathcal{S}$, where the right-side converges in weak sense.

Semiobservables are sometimes called unsharp observables, while ordinary observables, whose ranges are in the logic $\mathcal{U}$, are called sharp observables. Notice that the interval $\mathcal{E} = [0, e]$ endowed with the partial binary operation $\oplus$ defined by $f \oplus g$ is defined iff $f + g \leq e$ and in this case $f \oplus g = f + g$ becomes an interval effect algebra (Bennett and Foulis, 1996). We have the following characterization of elements of $\mathcal{U}$ in $\mathcal{E}$.

Lemma 2.5. For $x \in \mathcal{E}$, the following statements are equivalent:

(i) $x \in \mathcal{U}$, i.e., x is an extreme point in $[0,e]$.
(ii) $x^2 = x$.
(iii) The spectrum of x is contained in the set $\{0,1\}$.
(iv) $x \wedge (e - x) = 0$.
(v) $x \vee (e - x) = e$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows from the proof of Proposition 9.7 in Alfsen and Shultz (1976). (iii) $\Rightarrow$ (i): Assume that $x = \alpha y + (1 - \alpha) z$ with $y, z \in \mathcal{O}_1(L)$. Clearly, $m(x) = 0, 1$ iff $m(y) = m(z) = 0, 1$, respectively. From this we get $x\{1\} = y\{1\} \wedge z\{1\}$, $x\{0\} = y\{0\} \wedge z\{0\}$. Taking into account that by (iii), $x\{0\} = x\{1\}'$ $(= e - x\{1\})$, we have

$$x\{0\} = y\{0\} \wedge z\{0\} \leq y\{0\} \leq y\{0\} \vee z\{0\} \leq (y\{1\} \wedge z\{1\})'$$
$$= x\{1\}' = x\{0\}$$

hence $y\{0\} = z\{0\}$, i.e., $y = z$. This means that x is an extreme point. The equivalence (ii) $\Leftrightarrow$ (iv) has been proved in Greechie *et al.* (1995), and (iv) $\Leftrightarrow$ (v) follows by duality. ■

It turns out that with every instrument I (and hence with every dual instrument I^*) there is associated a semiobservable X which is defined by

$$X(E) = I^*(E)(e), \qquad E \in \mathcal{S} \tag{2.1}$$

An instrument I associated with a semiobservable X is called *X-compatible.* To every instrument, its associated semiobservable is uniquely defined, but there may be several instruments compatible with a given semiobservable.

Similarly as in the traditional Hilbert space approach (Lahti *et al.*, 1991), some important classes of instruments can be considered. The notions of c-ideal and d-ideal instruments were introduced in Pulmannová (1994, 1995). Before introducing our classification, we need the following definition. We recall that for a state m on a quantum logic L and $b \in L$, $m(b) \neq 0$, a function $p_m(a|b)$: $L \to [0,1]$ satisfying

(i) $p_m(a|b)$ is a state on L and $p_m(b|b) = 1$;
(ii) $a \in L,\ aCb \Rightarrow p_m(a|b) = m(a \wedge b)/m(b)$

where $a|b$ means that a, b are compatible, is called a *conditional state* (with respect to m and b). We say that L *admits conditional states with respect to a set of states* S if for any $m \in S$ and $b \in L$ with $m(b) \neq 0$ there is a conditional state $p_m(\cdot|b)$.

Projection lattices of von Neumann algebras with no I_2 factor as direct summand are well-known examples of logics with conditional states with respect to the set of all completely additive states. More generally, if (A, e) and (V, K) are in spectral duality, and (V, K) is a GL space and (A, e) its dual GM space, then the corresponding quantum logic $\mathcal{U}$ admits conditional states with respect to the set K. For the proof of the latter statement see Edwards and Rüttimann (1990).

Definition 2.6. Let (A, e) and (V, K) be in order and norm duality, and $(\Omega, \mathcal{S})$ a measurable space. An instrument I: $\mathcal{S} \to L^+(V(M), V(M))$ is called

(i) *repeatable* if, for all $E, F \in \mathcal{S}$ and $\rho \in K$,

$$\langle e, I(E)I(F)(\rho)\rangle = \langle e, I(E \cap F)(\rho)\rangle$$

(ii) *first* kind if, for the associated semiobservable X,

$$\langle X(E), \rho\rangle) = \langle X(E), I(\Omega)(\rho)\rangle$$

(iii) *ideal* if

$$y \in C(X), \langle y, \rho\rangle = 1 \Rightarrow \langle y, I(\Omega)(\rho)\rangle = 1$$

where $C(X)$ denotes the commutant of the semiobservable X, i.e., the set of all $y \in \mathcal{E}$ such that $yCX(E)$ for all $E \in \mathcal{S}$.

Assume that X is a discrete (sharp) observable with the spectrum $\{\omega_i\}_i$, $X(\{\omega_i\}) = b_i$. An instrument I compatible with X is

(iv) *d-ideal* if

$$\langle b_i, \rho\rangle = 1 \Rightarrow I(\{\omega_i\})(\rho) = \rho, \qquad \rho \in K, \quad \forall i$$

(v) *c-ideal* if

$$a \in C(X) \cap \mathcal{E} \Rightarrow \langle a, I(\{\omega_i\})(\rho)\rangle = \langle b_i \wedge a, \rho\rangle \qquad \forall \rho \in K, \quad \forall i$$

(vi) *Lüders* on a logic $\mathcal{U}$ with conditional states if

$$\langle a, I(\{\omega_i\})(\rho)\rangle = \langle b_i, \rho\rangle p_\rho(a|b_i) \qquad \forall \rho \in K, \quad \forall i$$

The same arguments as in Pulmannová (1994) can be used to prove that repeatability condition (i) is equivalent to

$$\langle e, I(E)I(E^c)(\rho)\rangle = 0 \tag{2.2}$$

and condition (ii) can be rewritten as follows:

$$\langle e, I(E)I(E^c)(\rho)\rangle = \langle e, I(E^c)I(E)(\rho)\rangle \tag{2.3}$$

for all $E \in \mathcal{S}$ and all $\rho \in K$.

We note that the ideality condition (iii) was chosen to represent a kind of minimal disturbance of the measured system caused by a measurement. Condition (iii) is equivalent to

$$a \in C(X) \cap \mathcal{U}, \qquad \langle a, \rho\rangle = 1 \Rightarrow \langle a, I(\Omega)(\rho)\rangle = 1 \tag{2.4}$$

Indeed, for any $y \in \mathcal{E}$, $P_{\{1\}}(y)$ is the unique element of $\mathcal{U}$ (an analogue of a range projection in von Neumann algebras) such that $\langle y, \rho\rangle = 1$ if and only if $\langle a, \rho\rangle = 1$ $(\rho \in K)$.

A repeatable instrument is always first kind, indeed,

$$\begin{aligned}\langle X(E), \rho\rangle &= \langle e, I(E)(\rho)\rangle \\ &= \langle e, I(\Omega)I(E)(\rho)\rangle = \langle e, I(E)I(\Omega)(\rho)\rangle \\ &= \langle I_E^*(e), I(\Omega)(\rho)\rangle = \langle X(E), I(\Omega)(\rho)\rangle\end{aligned}$$

The converse need not hold; in general, a first kind instrument need not be repeatable. An illustrative example has been introduced in Busch *et al.* (1991, 1996) as follows. Let a simple POV measure X be defined on the two-point value set $\{1, 2\}$ with $X\{i\} = A_i$. Now $0 \leq A_i \leq I$ and $A_2 = I - A_1$. Any X-compatible instrument is generated by any two state transformations φ_i $(i = 1, 2)$ with the property $\varphi_i(m)(1) = m(A_i)$, or $tr[\varphi_i T] = tr[TA_i]$ in terms of the density operators corresponding to the states m. Since $A_1A_2 = A_2A_1$, it follows that $tr[\varphi_1\varphi_2 T] = tr[\varphi_2\varphi_1 T]$ for any density operator T, so that this measurement is of the first kind. For example, $\varphi_i T = A_i^{1/2}\, T A_i^{1/2}$, $i = 1, 2$, are such. But such an instrument is repeatable if and only if A_i are projection operators, i.e., X is an ordinary observable (a PV-measure). It turns out that the notions of first kind instrument and repeatable instrument coincide in the context of ordinary observables; see Busch *et al.* (1991) for the Hilbert space approach and Pulmannová (1994) for a class of measurements on quantum logics.

Theorem 2.7. A first kind instrument which is compatible with a sharp observable is repeatable.

Proof. Let I be a first kind instrument compatible with an observable X. In terms of the dual instrument I^*, the first kind property can be expressed as

$$I^*(\Omega)(X(E)) = X(E) \tag{2.5}$$

for all $E \in \mathcal{S}$, or equivalently,

$$I^*(E)(X(E^c)) = I^*(E^c)(X(E)) \tag{2.6}$$

for all $E \in \mathcal{S}$. Making use of (2.5) and the additivity and positivity of I^*, one obtains

$$I^*(\Omega)(X(E)) = I^*(E)(X(E)) + I^*(E^c)(X(E)) = X(E)$$

and the same holds for $X(E)$ replaced by $X(E^c)$. Using (2.6), one then has

$$I^*(E)(X(E^c)) < X(E) \qquad \text{and} \qquad I^*(E)(X(E^c)) \leq X(E^c) \tag{2.7}$$

for all $E \in \mathcal{S}$. Since $X(E)$ and $X(E^c)$ are sharp elements of $\mathcal{E}$, by Lemma 2.5(iv) these inequalities imply that

$$I^*(E)(X(E^c)) = 0 \tag{2.8}$$

for all $E \in \mathcal{S}$, which is the equivalent repeatability condition (2.2). ■

An instrument I compatible with a semiobservable X will be called *range preserving* if $I^*\ (E)X(F)$ belongs to the range of X for every $E, F \in \mathcal{S}$. An element a of $\mathcal{E}$ is called *regular* if $a \not\leq \frac{1}{2}e$, $a \not\geq \frac{1}{2}\mathrm{e}$. A semiobservable is regular if its range consists only of regular elements of $\mathcal{E}$. We obtain the following statement.

Proposition 2.8. A first kind instrument compatible with a regular semi-observable is repeatable if and only if it is range-preserving.

Proof. Arguing in the same way as in the proof of Theorem 2.7, we arrive at (2.7). Now taking into account that $I^*(E)(X(E^c))$ belongs to the range of X and regularity of X implies that there is no nonzero element in its range satisfying inequalities (2.7), we obtain (2.8). ■

Proposition 2.9. An instrument I is ideal iff

$$a \in C(X) \cap \mathcal{U} \Rightarrow I^*(\Omega)(a) = a \tag{2.9}$$

Proof. By (2.4), ideality of I is equivalent to

$$a \in C(X) \cap \mathcal{U}, \qquad \langle a, \rho \rangle = 1 \Rightarrow \langle I^*(\Omega)(a), \rho \rangle = 1$$

Since the spectrum of I^* $(\Omega)(a) \leq I^*(\Omega)(e) = e$ is contained in $[0, 1]$, we have $\langle I^*(\Omega)(a), \rho \rangle = 1$ if and only if $\langle P_{\{1\}} (I^*(\Omega)(a)), \rho \rangle = 1$ $(\rho \in K)$. Since $a \vee a' = a + a' = e$, we get $P_{\{1\}}((I^*(\Omega)(a')) = P_{\{0\}},(I^*(\Omega)(a))$ and

$$e = a \vee a' \leq P_{\{1\}}(I^*(\Omega)(a)) \vee P_{\{0\}}(I^*(\Omega)(a))$$

hence the spectrum of $I^*(\Omega)(a)$ is contained in $\{0, 1\}$, i.e., it belongs to $\mathcal{U}$. From $a \leq I^*(\Omega)(a)$, and $a' \leq I^*(\Omega)(a') = e - I^*(\Omega)(a)$ we obtain $I^*(\Omega)(a) = a$.

The converse statement is clear. ■

Theorem 2.10. An ideal instrument which is compatible with a sharp observable is repeatable.

Proof. By Proposition 2.9, for every $E \in \mathcal{S}$ we have $I^*(\Omega)(X(E^c)) = X(E^c)$, hence $I^*(E)(X(E^c) \leq X(E^c)$. On the other hand, from $X(E) = I^*(E)(e)$ and $e = X(E) + X(E^c)$ it follows that $I^*(E)(X(E^c) \leq (X(E)$. Since the infimum of $X(E)$ and $X(E^c) = e - X(E)$ in $\mathcal{E}$ is 0, we obtain $I^*(E)(X(E^c)) = 0$, which is the repeatability condition (2.8). ■

Proposition 2.11. On any logic with conditional states, a Lüders measurement is d-ideal.

Proof. For a Lüders instrument I,

$$\langle b_i, \rho \rangle = 1, \qquad aCb_i \Rightarrow \langle a, I(\{\omega_i\})(\rho) \rangle = \langle b_i, \rho \rangle p_\rho(a|b_i)$$
$$= \langle a \wedge b_i, \rho \rangle = \langle a, \rho \rangle$$

Since the conditional state $p_\rho(\cdot|b_i)$ is uniquely defined by its values on $C(b_i)$ (where $C(b_i) = \{a \in L\text{: } aCb_i\}$), we have

$$I(\{\omega_i\})(\rho) = \langle b_i, \rho \rangle p_\rho(\cdot|b_i) = \rho$$

whenever $\langle b_i, \rho \rangle = 1$. ■

Proposition 2.12. A d-ideal measurement is ideal.

Proof. The d-ideality of I implies that for $\rho \in K$, and every i, $\langle b_i, \rho \rangle = 1 \Rightarrow \langle I^*(\{\omega_i\})(b_i), \rho \rangle = 1$. Hence for every i, $b_i \leq I^*(\{\omega_i\})(b_i)$, hence for $\forall \rho$,

$$\langle b_i, \rho\rangle \leq \langle P_{\{1\}}(I^*(\{\omega_i\})(b_i)), \rho\rangle$$
$$\leq \langle I^*(\{\omega_i\})(b_i), \rho\rangle \leq \langle I^*(\Omega)(b_i), \rho\rangle$$

Summing over i and taking into account that $\vee b_i = e$, we derive that $b_i = I^*(\Omega)(b_i)$ for every i.

Now let $a \in C(X) \cap \mathcal{U}$. Then $\langle a \wedge b_i, \rho\rangle = 1$ implies $\langle b_i, \rho\rangle = 1$, hence by d-ideality, $I(\{\omega_i\})(\rho) = \rho$, and hence $\langle P_{\{1\}}(I^*(\{\omega_i\})(a \wedge b_i)), \rho \rangle = 1$. This yields $a \wedge b_i \leq P_{\{1\}}(I^*(\{\omega_i\})(a \wedge b_i))$. Therefore, for every $\rho \in K$,

$$\begin{aligned}\langle a, \rho\rangle &= \sum_i \langle a \wedge b_i, \rho\rangle \\ &\leq \sum_i \langle P_{\{1\}}(I^*(\{\omega_i\})(a \wedge b_i)), \rho\rangle \\ &\leq \sum_i \langle I^*(\{\omega_i\})(a \wedge b_i), \rho\rangle \\ &\leq \sum_i \langle I^*(\{\omega_i\})(a), \rho\rangle \\ &\leq \langle I^*(\Omega)(a), \rho\rangle\end{aligned}$$

Similarly,

$$\begin{aligned}\langle a', \rho\rangle &\leq \sum_i \langle P_{\{1\}}(I^*(\{\omega_i\})(a' \wedge b_i)), \rho\rangle \\ &\leq \sum_i \langle I^*(\{\omega_i\})(a'), \rho\rangle \leq \langle I^*(\Omega)(a'), \rho\rangle\end{aligned}$$

But then

$$\begin{aligned}1 &= \langle a, \rho\rangle + \langle a', \rho\rangle \\ &\leq \sum_i \langle I^*(\{\omega_i\})(a), \rho\rangle + \sum_i \langle I^*(\{\omega_i\})(a'), \rho\rangle \\ &\leq \langle I^*(\Omega)(a), \rho\rangle + \langle I^*(\Omega)(a'), \rho\rangle = \langle I^*(\Omega)(1), \rho\rangle \leq 1\end{aligned}$$

which gives

$$\langle a, \rho\rangle = \sum_i \langle I^*(\{\omega_i\})(a)\rho\rangle = \langle I^*(\Omega)(a), \rho\rangle$$

and this yields the ideality condition (2.9). ■

In the next theorem, the equivalence of (i), (ii), and (iv) in the traditional approach has been proved in Lahti *et al.* (1991), and the equivalence of (i) and (iv) on von Neumann algebras in Luczak (n.d.). In Pulmannová (1995), there is a shortened proof of all the equalities for instruments on u-spectral logics.

Theorem 2.13. On the logics $\mathcal{U}$ with conditional states and for instruments compatible with sharp discrete observables, the following conditions are equivalent:

(i) ideal
(ii) d-ideal
(iii) c-ideal
(iv) Lüders

Proof. Equivalence of (iv) and (iii) can be proved as in Pulmannová (1994), (iv) $\Rightarrow$ (ii) follows by Proposition 2.11, and (ii) $\Rightarrow$ (i) follows by Theorem 2.12. It remains to prove (i) $\Rightarrow$ (iv). Let I be an ideal instrument for a discrete observable X. By Proposition 2.9, $I^*(\Omega)(a) = a$ for any $a \in C(X) \cap \mu$. By Theorem 2.10, I is repeatable. The repeatability condition (2.8) gives

$$a = I^*(\Omega)(a) = \sum_j I^*(\{\omega_j\})(\bigvee_i b_i \wedge a)$$

$$= \sum_j I^*(\{\omega_j\})(b_j \wedge a) = \sum_j I^*(\Omega)(b_j \wedge a)$$

and, also by repeatability, $I^*(\{\omega_j\})(a) = I^*(\{\omega_j\})(a \wedge b_j) = I^*(\Omega)(a \wedge b_j) = a \wedge b_j$; the last equality follows by Proposition 2.9. Hence for any j and $a \in C(X) \cap \mu$,

$$\langle a, I(\{\omega_j\})(\rho)\rangle = \langle a \wedge b_j, I(\{\omega_j\})(\rho)\rangle = \langle a \wedge b_j, \rho\rangle = p_\rho(a|b_j)\langle b_j, \rho\rangle$$

Owing to the uniqueness of conditional states, the equality

$$\langle a, I(\{\omega_j\})(\rho)\rangle = p_\rho(a|b_j)\langle b_j, \rho\rangle$$

holds for all $a \in \mathcal{U}$, hence I is Lüders. ∎

ACKNOWLEDGMENT

This work was supported by grant no. 4033/97 from the Slovak Academy of Sciences.

REFERENCES

Abbati, M., and Manià, A. (1984). Quantum logic and operational quantum mechanics, *Reports on Mathematical Physics,* **19**, 383–406.

Alfsen, E. M. (1971). *Compact Convex Sets and Boundary Integrals*, Springer-Verlag, Berlin.

Alfsen, E. M., and Shultz, F. W. (1976). Non-commutative spectral theory, *Memoirs of the American Mathematical Society,* **6**, No. 172.

Berberian, S. (1966). *Notes on Spectral Theory*, Van Nostrand, Princeton, New Jersey.

Bennett, M. K., and Foulis, D. J. (1996). Interval and scale effect algebras, *Advances in Appl. Math* **19**, 200–215.

Busch, P., Lahti, P., and Mittelstaedt, P. (1991). *Quantum Theory of Measurement*, Springer-Verlag, Berlin.

Busch, P., Lahti, P., and Mittelstaedt, P. (1996). *Quantum Theory of Measurement*, 2nd ed., Springer-Verlag, Berlin.

Davies, E. B. (1976). *Quantum Theory of Open Systems*, Academic Press, London.

Edwards, C. M., and Rüttimann, G. T. (1990). On conditional probability in GL spaces, *Foundations of Physics,* **20**, 859–872.

Lahti, P., Busch, P., and Mittelstaedt, P. (1991). Some important classes of measurements and their information gain, *Journal of Mathematical Physics,* **32**, 2770–2775.

Luczak, A. (n.d.). On ideal measurements and their corresponding instruments on von Neumann algebras, preprint.

Greechie, R. J., Foulis, D. J., and Pulmannová, S. (1995). The center of an effect algebra, *Order,* **12**, 91–106.

Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.

Pulmannová, S. (1980). Semiobservables on quantum logics, *Mathematica Slovaca,* **30**, 419–432.

Pulmannová, S. (1993). Boolean powers and quantum measurement, Reports on *Mathematical Physics,* **32**, 235–250.

Pulmannová, S. (1994). Quantum measurements and quantum logics, *Journal of Mathematical Physics,* **35**, 1555–1572.

Pulmannová, S. (1995). A quantum logics description of some ideal measurements, in *Quantum Communications and Measurement*, V. P. Belavkin, O. Hirota, and R. L. Hudson, eds., Plenum Press, New York.

Pulmannová, S. (n.d.). Quantum logics and convex spaces, preprint.

Rüttimann, G. T. (1985). Expectation functionals of observables and counters, *Reports on Mathematical Physics,* **21**, 213–222.

Varadarajan, V. S. (1985). Geometry of Quantum Theory, Springer-Verlag, Berlin.